Combining Loads from Random and Harmonic Excitation Using the Monte Carlo Technique

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Introduction

THE determination of the stresses or loads in a structure undergoing simultaneous harmonic and random excitation is a subject of some dispute in industry. In general, the dynamic response analyses of structures for random and harmonic cases are performed separately, and the results are then combined to obtain a design load. One example of such a structure is a component of a rocket engine. There can be several sources that produce random excitation, including the gas generator and main combustion chamber. These excitations are characterized by a power spectral density (PSD) spectrum of accelerations, which are frequently measured and scaled from a similar engine. These accelerations are used in a base excitation random response analysis, which results in PSD response spectra at any desired location. The mean of these spectra, defined simply as a load (although it can be in any desired quantity), is zero, and because the process can be assumed to be Gaussian,[1] the rms of the response is defined to be equal to one standard deviation σ. Simultaneously, the turbomachinery in the engine generates large harmonic loads due to the unavoidable unbalance in their rotors. The frequency of the harmonic excitations are at the shaft speed of the turbines and their multiples. The excitation from this sinusoidal loading, which can also be determined by previous acceleration measurements, is applied independently in a frequency response analysis, which generates the amplitude of a sinusoidal response for any desired location for a specified range of harmonic excitation frequencies.

The combination of random loads has been a subject of extensive research, in particular where quantifying the amount of error in the loads equations is concerned.[2] Detailed study of the use of similar equations for the combination of random and harmonically generated loads has not been performed, however. Perhaps because of this lack of examination, an industry standard loads combination equation has not been agreed on. There are, therefore, several different methodologies that have been applied by various users. They are all based on the method used for obtaining design loads for purely random environments, where standard practice is to use the $3 * \sigma_{\text{ran}}$ value for the design load. This value will exceed the response 99.86% of the time because the distribution is assumed to be Gaussian. The most frequently used method for random and harmonic loading, referred to here as the standard method, simply uses the sum of the amplitude of the sine load $A_{\sin}$ and three times the rms of the random load σ_{ran} (Ref. 3):

$$\text{design load} = A_{\sin} + 3\sigma_{\text{ran}} \tag{1}$$

This technique yields unnecessarily overconservative values because it assumes that the sine peak value always occurs simultaneously with the peak random value, which is unrealistic. Another method frequently applied is to multiply three times the square root of the sums of the squares (SRSS method) of the rms of both the random and the sines[4]:

$$\text{design load} = 3\sqrt{(\sigma_{\sin})^2 + (\sigma_{\text{ran}})^2} \tag{2}$$

where $\sigma_{\sin}$, the rms of the sine wave, is equal to the sine amplitude divided by $\sqrt{2}$. This method is also unrealistically overconservative because it treats the sine as if it had a normal probability distribution, which has a much larger range than a sine function. Neither of these closed-form methods produces a design load associated with a particular probability level.

There is considerable disagreement about whether the $3\sigma_{\text{ran}}$ should be used for the random loading-only case because the peaks that can cause fatigue fit a Rayleigh distribution better.[5] However, if this 99.86% level is chosen as the method for evaluating random dynamic loads, it would be consistent to evaluate the combination of sine and random loads in the same manner so that the design load would correspond with a specified probability of exceedence. A method is described hereafter that applies the Monte Carlo technique to generate this consistent design value for the combination of the loads resulting from these simultaneous excitations. This method produces a value that exceeds 99.86% of the responses for the combined excitation and is defined here as the equivalent 3σ value. Although this methodology is relatively straightforward, it has not been generally applied in industry and has not been discussed in the literature.

Loads Combination Using Probability Density Functions

Because a sine wave considered by itself is not a random signal, it is somewhat unclear how to combine harmonic analysis results with output from a random excitation analysis to obtain a design load. The harmonic signal can, however, be defined as a stationary random process when combined with an independent Gaussian random signal because its phase relationship ϕ with the random signal is random.[6] If the harmonic signal is defined as

$$p_{\sin} = A\sin(\omega t + \phi) \tag{3}$$

then the argument $(\omega t + \phi)$ is, therefore, a random variable with a uniform distribution over the range from $-\pi/2$ to $+\pi/2$. The probability density function of the sine function of this distribution is

$$f(y) = 1\big/\pi A\sqrt{1 - (y/A)^2} \tag{4}$$

The equivalent 3σ value can now be determined exactly by using the equivalent event technique for calculating the cumulative distribution functions (CDFs) of functions of random variables. For the function $z = x + y$, where x and y are random variables, the resultant CDF $F(z)$ is defined in terms of the probability density functions f_x and f_y as

$$F(z) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-y} f_x(x)\,\mathrm{d}x\right] f_y(y)\,\mathrm{d}y \tag{5}$$

If x represents the random load, it can be characterized by a normal distribution with mean zero and standard deviation equal to the rms from the random analysis. Substituting the normal probability density function and the probability density function shown in Eq. (4) into Eq. (5) results in

$$F(z) = \int_{-A}^{A}\left\{\frac{1}{\sigma_{\text{ran}}\sqrt{2\pi}}\int_{-\infty}^{z-y}\exp\left[\frac{-(x/\sigma_{\text{ran}})^2}{2}\right]\mathrm{d}x\right\} \times \frac{1}{\pi A\sqrt{1-(y/A)^2}}\,\mathrm{d}y \tag{6}$$

where the integral over y is evaluated only from $-A_{\sin}$ to $+A_{\sin}$ because the function is undefined outside of that range. A closed-form

Table 1 Comparison of integration and Monte Carlo methods

Case	A	σ	Z	$F(z)$ Integration method	$F(z)$ Monte Carlo	Difference, %
1	2	1	3	0.96553	0.96556	0.003
2	3	1	4	0.97218	0.97226	0.008
3	3	1	5.326	0.99846	0.99862	0.016

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Table 2 Comparison of methods used to combine sine and random loads

Sine amp	σ Random	Equivalent 3σ	Standard method	Difference from equivalent 3σ, %	SRSS	Difference from equivalent 3σ, %	Alternate A	Difference from equivalent 3σ, %
26.00	4.00	34.59	38.00	10	56.45	63	28.64	−17
97.00	14.67	128.28	141.00	10	210.42	64	106.51	−17
109.00	25.33	166.48	185.00	11	243.39	46	132.88	−20
99.00	46.33	210.05	238.00	13	251.84	20	170.65	−19
50.00	98.67	305.35	346.00	13	314.43	3	300.19	−2
64.00	109.33	351.62	392.00	11	354.99	1	334.19	−5
49.00	7.00	63.58	70.00	10	106.04	67	53.31	−16

solution for this integral cannot be obtained; however, the integral can be evaluated numerically for specific values of A and z for a $\sigma_{\text{ran}} = 1$ using the software code Mathematica.[7] Note that this solution is numerical and, therefore, not exact. Several solutions of the integral using this procedure are shown in Table 1.

Loads Combination Using Monte Carlo

The new method uses a Monte Carlo simulation to obtain this value more quickly and for any value of A, z, and σ_{ran}. The random load is first simulated by generating a vector of points $\{r\}$ that fall within a Gaussian (or normal) distribution of mean zero and standard deviation equal to the rms value resulting from the random analysis. This distribution can be expressed as $\{r\} \sim N(0.0, \sigma_{\text{random}})$. The vectors representing the results from the harmonic analysis are then generated. The method can account for analysis results from multiple independent harmonic excitations $\{y\}_i$, $i = 1, \ldots, n$. Because the frequencies of each sine wave are different and the relative phasing is unknown, each sine wave should be considered independently.[8] This method is only applicable for independent harmonic sources, such as excitations from different turbopumps. The loads coming from multiples of the primary shaft rotational speed for a single turbopump would be correlated and, therefore, not independent. For the independent case, a vector for each wave of the same length as $\{r\}$ is, therefore, created using the probability density function defined earlier, resulting in the following:

$$\{y\}_i = A_i \sin(\{x\}_i) \tag{7}$$

where $\{x\}$ is a random vector uniformly distributed between $-\pi/2$ and $+\pi/2$. The resulting vector $\{y\}$ will contain points that have the correct probability distribution of a sine wave, but are related randomly in phase. The summation of all of the harmonic signals and the random signal is performed for each sample to create a new vector $\{z\}$, which is equal to a sample set of total responses of the structure to the multiple simultaneous excitations:

$$\{z\} = \{r\} + \{y\}_1 + \{y\}_2 + \cdots + \{y\}_n \tag{8}$$

The CDF of $\{z\}$ is then generated, and a search procedure is performed to obtain the value from this CDF that is greater than 99.86% of the responses, defined earlier as the equivalent 3σ value. An example of a CDF of the combination of a random load with rms = 1 with a harmonic load of amplitude 1 and another harmonic load of amplitude 2 is shown in Fig. 1.

The accuracy of the Monte Carlo simulation is dependent on the number of samples selected. There are several methods for determining this accuracy; applying the method derived by Shooman,[9] the variation in the probability estimate p_f for a given number of samples n with a specified confidence is

$$p_f \pm k_{\alpha/2}\sqrt{p_f(1-p_f)/n} \tag{9}$$

For 95% confidence level with three-digit accuracy (resulting in a probability of 99.9%), the number of required samples is 124,689, which ran in 15 s (wall clock) on a Silicon Graphics O2 desktop workstation. If desired, the number of required samples can be reduced substantially if a more conservative, higher probability level is sought that allows a larger tolerance, or if other less stringent methods are used to determine the required number of samples. The accuracy of the analysis is also somewhat sensitive to the number

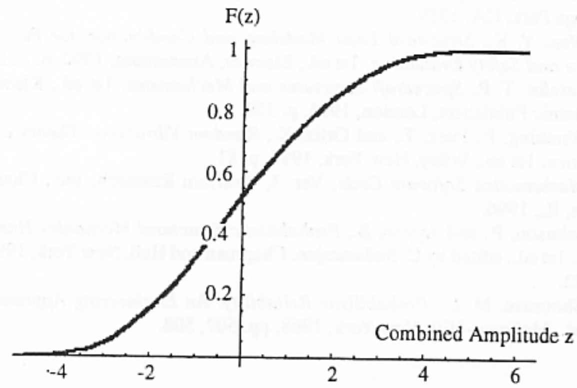

Fig. 1 CDF of combined loadings.

of simultaneous signals that are combined. For a combination of several signals, more consistent results were obtained using larger sample sizes than for a combination of just one random and one harmonic load.

Verification and Comparison

The Monte Carlo approach has been verified by comparing it with results obtained using the integration approach (Table 1). A sample size of 120,000 was used. As noted earlier, neither solution yields the exact answer, but the small error between the two serves to verify the Monte Carlo methodology.

Comparisons with the other methods for calculating the combined load were also carried out. In addition to the standard and SRSS techniques described earlier, another proposed approach was evaluated that attempts to reduce the conservatism and still maintain a closed-form solution. This method, called alternative A, takes the root sum square of the peak values:

$$\text{design load} = 3\sqrt{(A_{\text{sin}})^2 + (3\sigma_{\text{random}})^2} \tag{10}$$

Results for a large variety of combinations of random and sine loads (for a single sine wave) were calculated, and some are shown in Table 2. The complete set of results indicates that the standard approach can exceed the equivalent 3σ approach by up to 20%. The SRSS method also always exceeds the equivalent 3σ value, sometimes significantly, whereas alternative A generally yields values below the equivalent 3σ value and, hence, is unconservative.

Conclusion

The Monte Carlo technique has been applied for the rapid determination of loads on a structure that undergoes both random and harmonic excitations. This method produces a consistent value associated with any specified level of probability of exceedence. Application of this method can substantially reduce the amount of load used for the design of structures compared to traditional techniques. These design loads are especially critical for weight-sensitive aerospace structures. In addition, because of the speed of desktop computers, the analysis can be performed extremely easily and can result in values almost as quickly as the closed-form methods. The technique has been successfully applied to the determination of loads for 44 separate structural components in the FASTRAC engine program for 10 or more load cycles, resulting in large cost savings. One avenue of future work in this area is incorporating the effect of the correlation of the sine loads for harmonics of a single rotating shaft.

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